# Application of Optimality Criteria to Automated Structural Design

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**This paper presents a recursive design method for the minimum weight design of linear elastic redundant structures subject to multiple independent static loading conditions and with behavioral constraints on allowable element stresses and nodal displacements and constraints on design variables. This recursive method is based on the Kuhn-Tucker necessary conditions for a local optimum and gives, upon completion, a local optimum design. An iterative procedure is used to resize the structure until a design satisfying the Kuhn-Tucker necessary conditions is obtained. For resizing, it is necessary to identify the current near-active (critical) constraints and to use this data to construct the Kuhn-Tucker test. If the current design is not converged, then the information from the test is used to resize the design variables and improve the design. Each iteration or redesign requires only the solution of a set of linear algebraic equations equal in number to the number of currently active constraints. The method is used to design several well-known truss-type structures, and the results are shown to compare favorably with previous results obtained using mathematical programing algorithms and other optimality criteria methods.**

## I. Introduction

THE importance of efficient structural design on vehicle performance in aerospace applications has been a continuing motivation for research in optimum structural design, most of which has dealt with the minimum weight design of linear elastic structures. This problem is an inequality constrained minimization problem and may be investigated using mathematical programing algorithms.[1-3] The efficiency of the mathematical programing algorithms has been improved in the past decade[4-6] and now can be used to efficiently design structures with 100 independent design variables. However, present algorithms are still incapable of designing large structures with several hundred or more independent design variables.

The difficulty in obtaining optimum designs for large structures has resulted in the development of two approaches for determining near optimum designs. In the first approach, the problem size is reduced by relating the original set of design variables to a reduced set of generalized design variables via a linear transformation.[7] The original problem statement is replaced by an approximate inequality constrained minimization problem statement, and its optimum design is obtained using current mathmetical programing methods. The optimum design of the reduced problem is an approximation to the optimum design of the original problem, the degree of approximation depending on the prescribed linear transformation between the original and reduced set of design variables.

In the second approximate design method, the general inequality constrained minimization problem statement is replaced by an indirect and simpler problem statement.[8-11] A criterion related to structural behavior is postulated and the assumption is made that the minimum weight design is obtained when the structure is sized to satisfy the postulated criterion. The mathematical form of these criteria is similar to exact criteria developed by Prager and his co-workers[10] but they are typically applied to problems for which the criteria do not apply. In many cases, however, the use of such methods gives efficient, if not optimum, designs. For this reason, and because of their relative simplicity and computational efficiency compared to mathematical programing methods, the optimality criteria methods are widely used. The oldest and most widely used optimality criteria method is the fully stressing method, which is used for minimum weight design with constraints on allowable internal stresses. The inequality constrained problem statement is replaced by the indirect problem statement that "at the optimum design each element in the structure is fully stressed in at least one load condition." This criterion gives a simple iterative design formula (usually stress ratio), which can be efficiently applied to large structures. The fully stressed method is exact for determinate structures subject to one load condition but is approximate otherwise. Other optimality criteria are used for minimum weight design with displacement, frequency, and buckling constraints.[8-11]

Criteria based on the Kuhn-Tucker necessary conditions for a local optimum design have been proposed.[10] In application, however, the methods do not adhere to the Kuhn-Tucker conditions because: 1) elements governed by stress constaints are resized using stress ratio, and 2) in most cases the "envelope method" is used wherein resizing is accomplished by considering the most critical constraint for each element. The differences between designs obtained by stress ratio and/or "envelope methods" and the true optimum may be small as suggested in Refs. 10 and 11 or it may be substantial.

Both approximate methods (the reduction of the number of independent design variables through a linear transformation and the recasting of the problem using optimality criteria) give designs that approximate the optimum design to a degree difficult to assess. Therefore, research has continued for design methods which are 1) computationally efficient and 2) lead in general to local optimum designs.

An optimality criteria method satisfying both requirements was proposed by Terai[12] and is based directly on the Kuhn-Tucker necessary conditions for a local optimum design.[13] A simple recursive design formula was used to drive the design

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to a point in the design space satisfying the Kuhn-Tucker test and thereby giving a local optimum design.

The emphasis of Ref. 12 was not to directly obtain the optimum design using the recursive method but was, rather, to reduce the size of the design problem using generalized design variables.[7] The generalized design variables were obtained using the recursive method with only one type of behavioral constraint active at a time. Special forms of the Kuhn-Tucker based recursive method were derived for several types of behavioral constraints. Stress limited design problems were not studied using this recursive method because of the sensitivity of stress constraints to design variable changes. Rather, the stress ratio method was used to develop approximate optimum designs for this problem.

The purpose of the present study is to 1) extend the recursive method of Ref. 12 to stress limited problems and 2) obtain minimum weight designs for linear elastic redundant structures subject to multiple independent static loading conditions with mixed behavioral constraints and side constraints on the design variables. The method developed herein is capable of determining the subset of critical constraints at the optimum design from the set of total possible constraints. The structure is redesigned until this subset is obtained and the Kuhn-Tucker necessary conditions are satisfied.

The present method converges to a local optimum design, whereas previous (or "physical") optimality criteria methods in general do not. But, unlike the physical criteria, the present method requires the calculation of the derivatives of the active constraints and the solution of a set of linear algebraic equations equal in number to the number of active constraints for the resizing of the design variables. For resizing, mathematical programing methods require, in general, the solution of a set of linear algebraic equations equal in number to the number of design variables. Therefore, the present method is slightly less efficient than physical optimality criteria methods but more efficient than mathmetical programing methods, particularly for problems in which the number of final active constraints is smaller than the number of design variables. However, the results is Sec. III show the present method to be more efficient than mathematical programing methods even for large stress limited problems with a large number of design variables and active final constraints.

## II. The Mathematical Optimality Criterion and Redesign Rule

### A. Review of the Optimality Criterion

The minimum weight structural design problem is an inequality constrained minimization problem stated as follows: find a set of independent (linked) design variables $D_J, j=1.2, \ldots, N$, such that

$$g_k(D_j) \leq 0 \qquad k=1,2,\ldots,M \tag{1}$$

and

$$W(D_j) \rightarrow \min \tag{2}$$

where the objective function $W$ is the weight and the constraint functions $g_k$ are behavioral constraints and side constraints on the design variables.

The Kuhn-Tucker necessary conditions for a design to be a relative or local minimum are

$$\frac{\partial W}{\partial D_j} + \sum_k^M \lambda_k \frac{\partial g_k}{\partial D_j} = 0 \qquad J=1,2,\ldots,N \tag{3}$$

$$\lambda_k g_k = 0$$

$$\lambda_k \geq 0 \qquad k=1,2,\ldots,M \tag{4}$$

where $\lambda_k$ is the Lagrange multiplier associated with constraint $g_k$. If the constraint is noncritical $g_k<0$ and $\lambda_k=0$; if the constraint is critical $g_k=0$ and $\lambda_k \geq 0$.

The statement of the Kuhn-Tucker test may be simplified to read

$$\frac{\partial W}{\partial D_j} + \sum_k^{\bar{K}} \lambda_k \frac{\partial g_k}{\partial D_j} = 0 \qquad j=1,2,\ldots,N \tag{5}$$

where $g_k=0$ and $\lambda_k \geq 0$ for all $\bar{K}(\leq M)$ critical or active constraints.

The side constraints on design variables are

$$-D_j + D_{j\min} \leq 0 \qquad j=1,2,\ldots,N \tag{6}$$

$$D_j - D_{j\max} \leq 0 \qquad j=1,2,\ldots,N \tag{7}$$

where $D_{j\max}$ and $D_{j\min}$ are upper and lower bounds on $D_j$, respectively.

The active side constraints, Eqs. (6–7), can be separated from the active behavioral constraints and the Kuhn-Tucker necessary conditions for a local optimum can be written[12,15]

$$\frac{\partial W}{\partial D_j} + \sum_k^K \lambda_k \frac{\partial g_k}{\partial D_j} = 0 \qquad j \in J \tag{8}$$

$$\frac{\partial W}{\partial D_j} + \sum_k^K \lambda_k \frac{\partial g_k}{\partial D_j} \leq 0 \qquad j \in J_{\max} \tag{9}$$

$$\frac{\partial W}{\partial D_j} + \sum_k^K \lambda_k \frac{\partial g_k}{\partial D_j} \geq 0 \qquad j \in J_{\min} \tag{10}$$

where $J$ is the set of design variables such that $D_{j\min} < D_{j\max}$, $J_{\max}$ and $J_{\min}$ are the set of design variables constrained by maximum and minimum values, respectively, and $K$ is the set of active behavioral constraints.

The separated form of the Kuhn-Tucker necessary conditions may be rewritten[12]

$$I_j = -\frac{\sum_k^K \lambda_k \left(\frac{\partial g_k}{\partial D_j}\right)}{\left(\frac{\partial W}{\partial D_j}\right)} \quad \begin{bmatrix} =1, j \in J \\ \geq 1, j \in J_{\max} \\ \leq 1, j \in J_{\min} \end{bmatrix} \tag{11}$$

These relations are the criteria for minimum weight and are the basis for the redesign method used in the present study. If, for a given design, the set of $K$ active constraints is such that the corresponding Lagrange multipliers are positve and Eq. (11) is satisfied, then the design is a local optimum design. It is not necessary to prespecify the type of behavioral constraint in Eq. (11).

At any stage of design process, given the current design variables $D_j$, it is possible to determine the set of active or nearly active constraints and then calculate the corresponding Lagrange multipliers. The calculation of the multipliers can be accomplished several ways.[12,16] In the present study, an auxiliary function $\ell$ is formed

$$\ell(\lambda) = \sum_j^J \left[1 + \left(\sum_k^K \lambda_k \frac{\partial g_k}{\partial D_j}\right) \Big/ \frac{\partial W}{\partial D_j}\right]^2 \tag{12}$$

and is minimized by solving the set of equations

$$\frac{\partial \ell}{\partial \lambda_k} = 0 \qquad k=1,2,\ldots,K \tag{13}$$

for the Lagrange multipliers $\lambda_k$. Since, in general, there are more design variables than active constraints, this amounts to minimizing the mean square error which is, from Eqs. (11) and (12)

$$\ell = \sum_j^J (I_j - 1)^2 \tag{14}$$

**B. Resizing Rule**

Qualitatively, the Kuhn-Tucker optimality criterion of Eq. (11) implies that: 1) if $I_j = 1.0$ then $D_j$ is optimally sized; 2) if $I_j > 1.0$ and $D_j < D_{j\max}$ then $D_j$ must be increased to obtain an improved design; and 3) if $I_j < 1.0$ and $D_j > D_{j\min}$ then $D_j$ must be decreased to obtain an improved design. The resizing rule is therefore taken as a function of current values of $I_j$, i.e.,

$$D_j^{\alpha+1} = f[I_j(D_j^{\alpha})]D_j^{\alpha} \tag{15}$$

where the design variables at the $\alpha$th iteration are given and $f(1.0) = 1.0$.

In the present study the function is taken to be

$$f(I_j) = I_j^{1/2} \tag{16}$$

The resizing rule then becomes

$$D_j^{\alpha+1} = I_j^{1/2} D_j^{\alpha} \qquad D_{j\min} < I_j^{1/2} D_j^{\alpha} < D_{j\max} \tag{17}$$

together with auxiliary condtions which account for side constraints

$$D_j^{\alpha+1} = D_{j\min} \qquad I_j^{1/2} D_j^{\alpha} \leq D_{j\min} \tag{18}$$

and

$$D_j^{\alpha+1} = D_{j\max} \qquad I_j^{1/2} D_j^{\alpha} \geq D_{j\max} \tag{19}$$

The functions $I_j^{1/2}$, $j = 1, 2, \ldots, N$ are hereafter referred to as the design factors.

This resizing rule is exact for the simple case of statically determinate structures with weight functions linear in the design variables and subject to a single frequency or buckling constraint, or subject to a single load condition and a single stress or displacement constraint. The resizing rule is approximate for the design of statically indeterminate structures and/or multiple load condition problems. The use of the resizing rule of Eqs. (17-19) for these cases is equivalent to assuming that the internal force distribution does not change during resizing. This is the fundamental assumption of the stress ratio resizing rule for fully stressed design and, in this sense, the present method and the fully stressing method are similar.

Resizing the structure may cause the set of current critical constraints to change (constraint switching). Therefore, the set of active constraints must be checked after each resizing and updated as necessary. In practice, as soon as the set of active constraints stabilizes, the algorithm converges rapidly to a final design.

The resizing rule (Eq. 17) was not used in Ref. 12 for stress-limited design because: 1) the resizing rule is valid only near the optimum and is approximate elsewhere and 2) member stress constraints are sensitive to changes in the individual design variables. This sensitivity causes a marked switching in the set of active constraints and may lead to infeasible designs in the initial design stages. In this work, the problems were resolved by amending the resizing procedure for stress limited problems by: 1) imposing move limits on the design variable changes; and 2) scaling to the most active constraint when infeasible designs are obtained, that is,

$$\bar{D}_j^{\alpha+1} = \bar{R} D_j^{\alpha+1} \tag{20}$$

where $\bar{R}$ is a scaling factor equal to the ratio of the developed value to the limiting value of the most critical constraint.

**C. Redesign Algorithm**

A chart of the general operation of the present optimality criteria method during each iteration or redesign cycle is shown in Fig. 1. This flow chart does not include the identification of current active constraints before entering or the scaling operation after exiting.

The functional $\ell$ is minimized to solve for the Lagrange multipliers and the design factors $I_j^{1/2}$. Uniform move limits $\pm\Delta$ are imposed on the design factors and a trial resizing performed. (In addition, for stress limits only, a fully stressing move is made on those design variables for which the corresponding design factors are less than zero. This move is useful in quickly identifying minimum area design variables).

It is necessary to modify the functional $\ell$ and recalculate $\lambda_k$ and $I_j^{1/2}$ if, in the trial resizing, 1) new side constraints are identified, or 2) negative multipliers are calculated. When a new side constraint is identified, it is necessary to delete the design variable from the sum in Eq. (13) over the variables

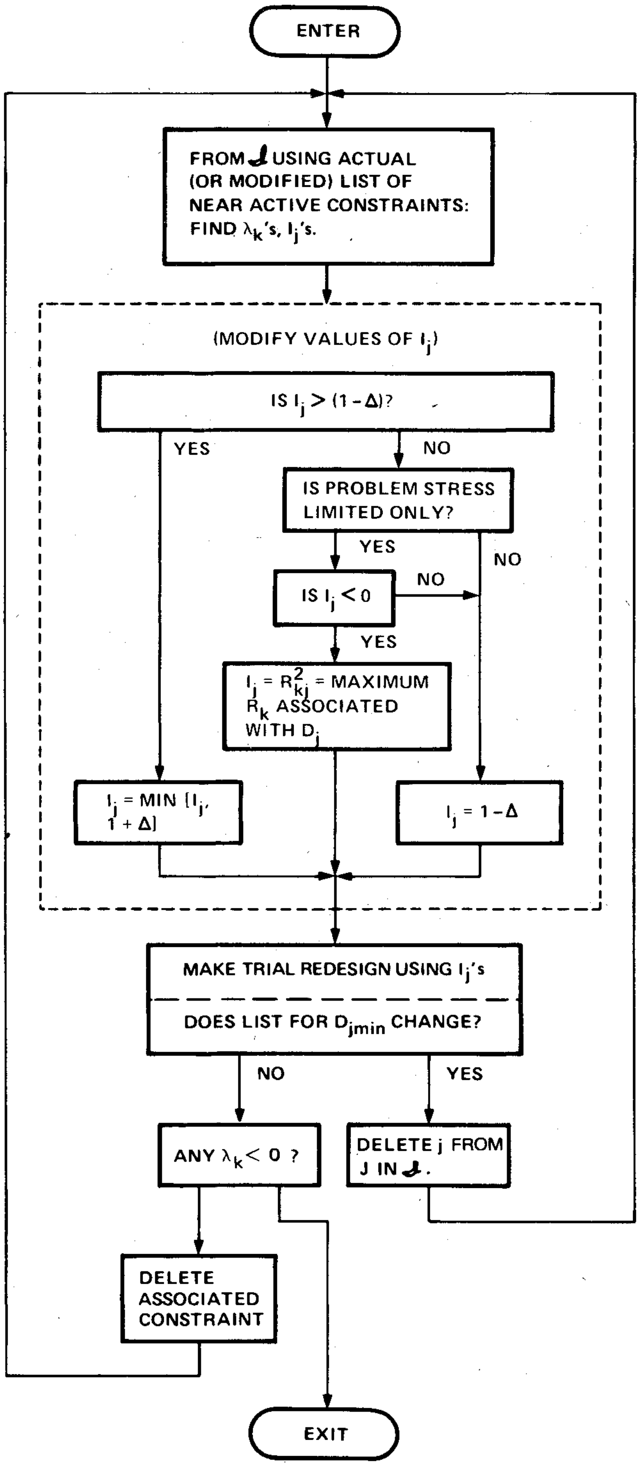


**Fig. 1 Flow chart for redesign algorithm.**

contained in the set $J$. Negative multipliers are typically obtained in the early cycles of redesign. When a negative multiplier is obtained, it is necessary to delete the corresponding constraint from the sum in Eq. (13) over the set of $K$ active behaviorial constraints.

Both the deletion of design variables associated with side constraints and the deletion of behavioral constraints associated with negative multipliers may require, within each redesign cycle, one or more iterations to obtain a stable list of side constraints and active behavioral constraints. When the list stabilizes, the resized structure is analyzed and scaled and the algorithm reentered.

**D. Computational Aspects**

Behavioral constraints are identified as active and governing the redesign if

$$1.0-\beta\leq R_k\leq 1.0 \tag{21}$$

where $R_k$ is the "response ratio"[6] of the $k$th constraint and $\beta$ is a preselected constraint buffer parameter. The identification of active constraints and the final weight obtained are sensitive to the value of the constraint buffer. For both stress limited and stress and displacement limited problems, the constraint buffer is typically in the range $0.04\leq\beta\leq 0.10$. Larger values of $\beta$ allow early identification of active constraints, but usually result in heavy final designs because the final "active" constraint criticalities are low. Smaller values of $\beta$ usually result in lighter final designs because of the higher criticalities of the final near-active constraints. (Other methods to solve for the Lagrange multipliers may minimize the need for constraint criticality buffers for problems with a relatively small and stable set of active constraints but may not be applicable to large stress limited problems.[15]

Move limits are imposed on the design factors because of the approximations inherent in the redesign rule, Eq. (17). For stress limited problems, the move limits must be relatively small and typically in the range $0.04\leq|\nabla|\leq 0.08$. For displacement limited problems (or mixed stress and displacement limited problems in which the displacement limits predominate) the move limits may be greater and are typically $|\Delta|=0.25$. Displacement constraints are usually less sensitive than stress constraints to individual design variable changes and the set of active displacement constraints is usually smaller than the set of active stress constraints. Therefore, the set of active constraints in displacement limited problems is relatively stable and large design variable changes are possible.

Computational experience indicates that the following default values for the two preselected design parameters $\beta$ and $\Delta$ are usually adequate for obtaining near optimum designs: 1) for stress limited problems $\beta=0.08$, $|\Delta|=0.08$; 2) for stress- and displacement limited problems $\beta=0.10$, $|\Delta|=0.25$. The final designs for all the example problems of Sec. III (with the exception of the 63 bar truss with stress and relative displacement constraints) were first obtained using the default values previously given.

It is usually possible to improve the final designs obtained with the default design parameters by reducing the value of the constraint criticality buffer $\beta$, although more iterations may be necessary for convergence. For stress and displacement limited problems the buffer $\beta$ can be chosen independently of the move limit $\Delta$. However, because of the sensitivity of the stress constraints in stress limited problems, it is necessary to reduce the value of the move limits as well as the criticality buffer.

## III. Examples

The minimum weight designs for several well-known problems (the 10, 25, and 63 bar trusses) have been obtained using a computer code for the redesign algorithm shown in Fig. 1. (Additional stress limited designs for the 3, 9, 25, 63, and 72 bar trusses are given in Ref. 14). All computations were performed on the UCLA Campus Computing Network (CCN) IBM 360/91 computing system using single precision arithmetic; the FORTRAN G compiler option was used unless noted otherwise. All the problems were initiated from a uniform design and all final results are converged ($\sqrt{I_j}\equiv 1.0$ for all nonminimum area variables; $\sqrt{I_j}\leq 1.0$ for minimum area variables) unless specified otherwise. Comparison of CPU times required for execution of the present method with execution times of previous methods are not made except for the 63 bar truss because of the differences in computers and/or compilers used.

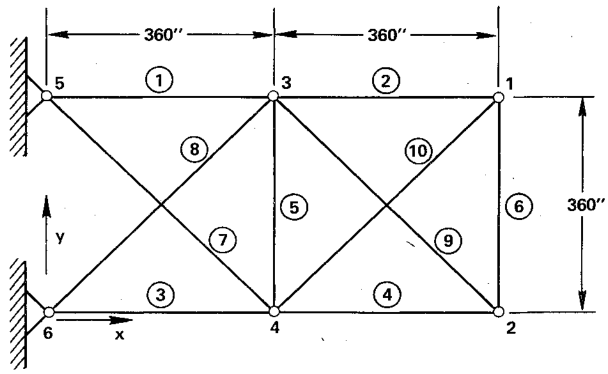


**Fig. 2 Planar ten bar cantilever truss.**

**A. Ten Bar Planar Truss**

The structural configuration is shown in Fig. 2. The material is aluminum with elastic modlus $E=10\times 10^3$ ksi and density $\rho=0.1$ pci. No design variable linking is specified; therefore, this problem has 10 independent design variables. A minimum side constraint $=0.1$ in.$^2$ is imposed on all problems; maximum side constraints are not imposed.

*1. Stress Limits Only, Single Load Condition*

Three stress limited problems are considered. In each problem, the loading consists of two 100 K loads applied in the negative $y$ direction at nodes 2 and 4 (Fig. 2). In the first problem the member stress limits are $\pm 25$ ksi. Problems two and three are identical except that the stress limits of member nine are increased to $\pm 30$ ksi and $\pm 50$ ksi, respectively.

These problems were previously studied using physical optimality criteria methods[10] and mathematical programing methods[6] and were found to require a relatively large computational effort as compared to other problems of this size. In fact, the increase in the stress limits of member nine from $\pm 30$ ksi to $\pm 50$ ksi causes other optimality criteria iteration methods to yield converged but nonoptimum designs. The difficulty of this problem is reflected in the present method by the large number of cycles for convergence.

The results of the present study are given in Table 1. Comparison of final results with previous results is made in Table 2. For the first two problems, the present algorithm converges quickly to near optimum designs with weights within 2% of the previously published results. The optimum designs for both problems have six active stress constraints (in members 1, 3, 4, 7, 8, and 9) and four active side constraints (in members 2, 5, 6, and 10). The results for problem 1 were obtained in 10 iterations with the stress limited default parameters $\beta=0.08$, $|\Delta|=0.08$. Initial results ($W=1{,}620$ lb) for problem 2 required 14 iterations with the default parameters; final results ($W=1{,}582$ lb) were obtained in 14 iterations with $\beta=0.05$ and $|\Delta|=0.05$.

The method converges slowly for the third problem. Increasing the allowable stress of member nine to $\pm 50$ ksi results in a design with eight active stress constraints (member 1-8, and 10) and three active minimum area constraints (members 2, 5, and 6) in the final design. Therefore, there are 11 active constraints in a design problem with 10 independent design variables. However, the local symmetry peculiar to this

Table 1 Final designs for 10 bar planar truss, stress limits only (Sec. III, A1)

| Truss Member No. | Final cross-sectional areas (in.$^2$) ±25,000 psi, Bar 9 | ±30,000 psi, Bar 9 | ±50,000 psi, Bar 9 |
|---|---|---|---|
| 1 | 8.141 | 8.227 | 7.920 |
| 2 | 0.100 | 0.100 | 0.100 |
| 3 | 8.343 | 8.302 | 8.164 |
| 4 | 3.951 | 4.009 | 3.910 |
| 5 | 0.100 | 0.100 | 0.100 |
| 6 | 0.100 | 0.100 | 0.100 |
| 7 | 5.768 | 5.743 | 5.859 |
| 8 | 5.760 | 5.785 | 5.514 |
| 9 | 5.570 | 4.717 | 3.776 |
| 10 | 0.100 | 0.100 | 0.141 |
| Weight, Lb | 1622 | 1582 | 1509 |
| Iterations (analyses) | 10(11) | 14(15) | 114(115) |

problem required that the areas at bars 2 and 6 be equal (and minimum) and the area of bar 10 be $\sqrt{2}$ the area of bars 2 and 6. Thus, there are only eight independent design variables and eight independent active constraints (six stress and two minimum area constraints) in the final design. The final results ($W = 1,509$ lb) were obtained in 114 iterations with the stress limited default parameters and are within 1% of the results of Ref. 6.

The fully stressing design method[10] converges in 14 iterations but to a nonoptimum design with $W = 1,725$ lb. The present method gives a weight of 1,642 lb after 14 iterations (and 1,610 lb after 14 iterations with $\beta = 0.04$ and $|\Delta| = 0.05$). It is noted that an inspection of the trends of the design factors prior to convergence reveals the local symmetry of bars 2 and 6 and suggests that bars 2, 6, and 10 are constrained by both stress limits and minimum area.

Table 2 Comparison of results, 10 bar planar truss stress limits only (Sec. III, A1)

| ($\sigma_9$) allowable | Weight (lbs), number of analyses: Present results | Ref. 10 (Phys. Optimality) | Ref. 6 (Math. Prog.) |
|---|---|---|---|
| ±25 ksi | 1622,11 | 1593,16 | 1593,16 |
| ±30 | 1582,15 | 1545,23 | 1545,16 |
| ±50 | 1509,115 | 1725,14 | 1498,16 |
| | 1610,14[a] | | |

[a]Not converged.

*2. Stress and Displacement Constraints, Single-Load Condition*

Two stress- and displacement limited problems are considered. In each problem, the member stress limits are ±25 ksi and the nodal displacement limits are ±2.0 in. In the first problem, the single-load condition consists of two 100 K loads applied in the negative $y$-direction at nodes 2 and 4. In the second problem, the loading consists of two 150 K loads applied in the negative $y$-direction at nodes 2 and 4 and two 50 K loads applied in the positive $y$-direction at nodes 1 and 3. Although the net loads on the structure are the same for both problems, identification of the final active constraints is more difficult for the second load case.

The results of the present study are given in Table 3 and are compared to previous results in Table 4. Initial results for problem 1 ($W = 5,116$ lb) required 26 iterations using the displacement limited default parameters $\beta = 0.10$, $|\Delta| = 0.25$; final results were obtained in 14 iterations with $\beta = 0.04$, $|\Delta| = 0.25$ and are essentially identical to previously reported results. This problem has three active behavioral constraints (one stress and two displacement constraints) and three minimum area constraints in the final design.

Final results for problem 2 required 12 iterations using the displacement limited default design parameters. The converged results ($W = 5,060$ lb) are 8% greater in weight than the established optimum.[6] This problem has three active behavioral constraints (two stress and one displacement constraint) and three minimum area constraints in the final

Table 3 Final designs for 10 bar planar truss, stress and displacement limits (Sec. III, A2)

| Truss Member No. | Final cross-sectional areas (in.$^2$) Problem 1 | Problem 2 |
|---|---|---|
| 1 | 30.500 | 25.813 |
| 2 | 0.100 | 0.100 |
| 3 | 23.290 | 27.233 |
| 4 | 15.428 | 16.653 |
| 5 | 0.100 | 0.100 |
| 6 | 0.210 | 2.024 |
| 7 | 7.649 | 12.776 |
| 8 | 20.980 | 14.218 |
| 9 | 21.818 | 22.137 |
| 10 | 0.100 | 0.100 |
| Weight, Lb | 5080.0 | 5059.7 |
| Iterations (analyses) | 14(15) | 11(12) |

Table 4 Comparison of results, 10 bar planar truss, stress and displacement limits (Sec. III, A2)

| | Weight (lb), number of analyses: Present Results | Ref. 9 (Phys. Optimality) | Ref. 12 (Variable Reduction) | Ref. 6 (Math. Prog.) |
|---|---|---|---|---|
| Problem 1 | 5080.0, 15 | 5084.9, 26 | 5077.6, 21[b](33)[a] | 5096.7, 13 |
| Problem 2 | 5059.7, 12 | 4895.6, 13 | 4708.6, 20[b](29)[a] | 4676.9, 11 |

[a]Total analyses, including generalized variable generation and nonlinear programing search.
[b]Total analyses including approximization concepts.

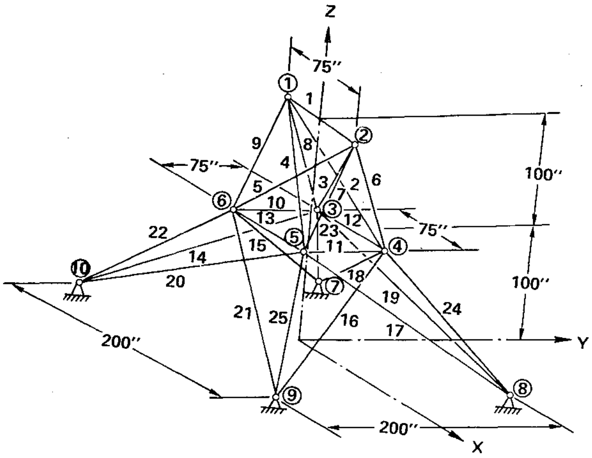

Fig. 3 Twenty-five bar space truss.

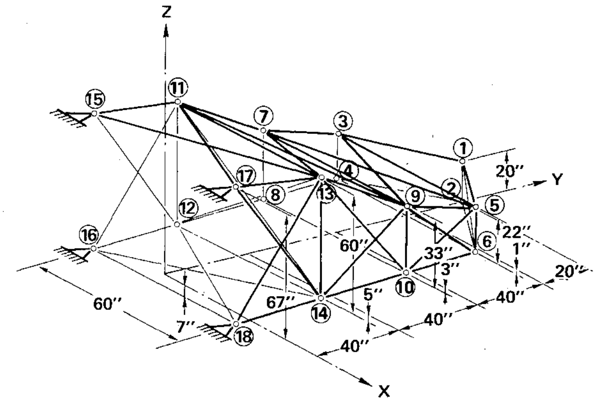

Fig. 4 Truss idealization of wing carry through structure. Note: For the sake of clairty, not all elements are drawn in this figure.

design. The response ratio of the single active displacement constraint in the final design is only 91.5%. A lighter weight feasible design ($W=4{,}850$ lb) was obtained during the redesign (after nine cycles) with one active stress and one active displacement constraint. Thus, this design was not converged in the sense that the proper set of active constraints was not identified, but the response ratio of the active displacement constraints was 97.5%. Since individual displacement constraints have a large effect on truss weight (as compared to the effect of individual stress constraints) it is not surprising that the unconverged design would be lighter than the fully converged design.

**B. Twenty-Five Bar Space Truss, Stress and Displacement Constraints**

The structural configuration is shown in Fig. 3. The material is aluminum with $E=10\times10^3$ ksi and $\rho=0.1$ pci. The 25 members are linked to eight independent design variables. A minimum area constraint $=0.1$ in.$^2$ is imposed on the design variables; maximum limits are not imposed. (In the literature this problem usually has a minimum area constraint $=0.01$ in.$^2$ The increase in the limit to 0.1 in.$^2$ has no practical effect on final weight, although it influences the rate of convergence; see below).

The truss is subject to two independent load conditions.[6] The tensile stress limit is 40 ksi for all members, although the compressive stress limits vary for each member.[6] The nodal displacement limits are $\pm 0.35$ in. The results required 10 iterations with the displacement limited default parameters and are compared to previous results in Table 5. The present method converges to a design that is 2% heavier than the established minimum. The final design has three active behavioral constraints (one stress and two displacement constraints) and three active minimum area constraints (design variables 1, 4, and 5; members 1, 10, 11, 12, and 13).

The present algorithm does not converge, even after 35 iterations to a design identically satisfying the Kuhn-Tucker necessary conditions when the minimum area constraint is reduced to 0.01 in.$^2$ The lack of final convergence is due solely to the inability of the method to identify the minimum area constraint of design variable 1 (member 1). The remaining minimum area constraints and the three behavioral constraints are, however, identified. The design factors $\sqrt{I_j}$ $=1.0\pm3\%$ and the weight is within 2% of the established minimum. The lack of final convergence points out the sensitivity of the algorithm to minimum area identification and the need for realistic choices of minimum area constraints.

**C. Sixty-Three Bar Space Truss (Idealization of Wing Carry-Through Box)**

The structural configuration is shown in Fig. 4. The material is titanimum with $E=16\times10^3$ ksi and $\rho=0.16$ pci. No design variable linking is specified; therefore, this problem has 63 independent design variables. A minimum area constraint $=0.01$ in.$^2$ is imposed on the design variables; maximum limits are not imposed.

*1. Stress Limits Only, Two Load Conditions*

The member stress limits are $\pm 100$ ksi. The truss is subject to two independent load conditions.[6] A weight of 5,145 lb was obtained after 15 analyses using the stress-limited default design parameters. An improved result with $W=5026$ lb was obtained after 12 analyses with $\beta=0.06$, $|\Delta|=0.05$. These results detailed in Table 6 compare favorably with the stress ratio design[10] of 5,034 lb after 50 analyses and the design obtained using mathematical programing[6] with $W=4977$ lb after 14 analyses.

It is clear from Table 7 that the convergence of the present method is not monotonic. The fluctuation in the number of active constraints is primarily due to 1) the large number of active constraints in the final design; 2) the sensitivity of member stresses to changes in member areas; and 3) the approximations inherent in the resizing rule. The final designs in Refs. 6 and 10 are essentially fully stressed. The final design in Ref. 6 has 58 active stress constraints and five active minimum area constraints. The final design of the present method has 44 active stress constraints and nine active minimum area constraints; this design is not strictly converged since the Kuhn-Tucker necessary conditions are not identically satisfied. However, the design factors of 42 of the 54 nonminimum area bars are converged to $\pm5\%$ of 1.0 and the remaining design factors are converged to $\pm10\%$ of 1.0. Continued redesign would not lead to large changes in variable sizes or final weight.

The final results for this problem using the present method required 18 CPU sec for execution on the IBM 360/91 using the FORTRAN H compiler. The method of Ref. 6 required

Table 5 Comparison of results, 25 bar space truss, stress and displacement constraints (Sec. III B)

| | Weight (lb), number of analyses | | |
|---|---|---|---|
| Present Results | Ref. 9 (Phys. Optimality) | Ref. 12 (Variable reduction) | Ref. 6 (Math. Prog.) |
| 553.4, 10 | 545.5, 7 | 551.6, 17(21) | 545.2, 10 |

Table 6 Final designs for 63 bar wing carry through box, stress constraints only (Sec. III C1)

| Truss Member no. | Final cross-section areas (in.$^2$) | Truss member no. | Final cross-section areas (in.$^2$) |
|---|---|---|---|
| 1 | 38.36 | 33 | 5.26 |
| 2 | 36.53 | 34 | 5.85 |
| 3 | 52.66 | 35 | 5.26 |
| 4 | 54.53 | 36 | 3.27 |
| 5 | 25.24 | 37 | 3.36 |
| 6 | 28.55 | 38 | 3.33 |
| 7 | 17.87 | 39 | 3.44 |
| 8 | 20.73 | 40 | 5.31 |
| 9 | 25.14 | 41 | 5.19 |
| 10 | 27.12 | 42 | 16.03 |
| 11 | 7.59 | 43 | 18.42 |
| 12 | 9.16 | 44 | 11.69 |
| 13 | 24.08 | 45 | 14.09 |
| 14 | 20.53 | 46 | 11.82 |
| 15 | 4.16 | 47 | 6.45 |
| 16 | 2.62 | 48 | 11.90 |
| 17 | 37.39 | 49 | 13.85 |
| 18 | 37.58 | 50 | 7.47 |
| 19 | 0.01 | 51 | 7.65 |
| 20 | 0.01 | 52 | 5.35 |
| 21 | 0.38 | 53 | .20 |
| 22 | 0.18 | 54 | 3.56 |
| 23 | 0.01 | 55 | 9.45 |
| 24 | 1.11 | 56 | 4.30 |
| 25 | 0.34 | 57 | .47 |
| 26 | 2.93 | 58 | .01 |
| 27 | 1.08 | 59 | .01 |
| 28 | 4.72 | 60 | .01 |
| 29 | 0.85 | 61 | .01 |
| 30 | 3.12 | 62 | .01 |
| 31 | 3.36 | 63 | .01 |
| 32 | 5.57 | Weight (lb) | 5026.4 |
| | | Iterations (Analyses) | 11(12) |

approximately 48 CPU sec of execution time on the same IBM 360/91 computer and the FORTRAN H compiler for a 1% overweight design ($W=5{,}045$ lb) using default design parameters. Subsequent results in Ref. 6 with design parameters selected to speed convergence required approximately 36 CPU sec for a 1% overweight design ($W=5{,}025$ lb). Thus, the present optimality criteria method retains a computational efficiency advantage over mathematical programing methods even when the number of design variables is larger than the number of degrees of freedom of the truss and the number of final active constraints is not small compared to the number of design variables. In part, this efficiency advantage is due to the small number of active constraints in the early stages of the design process (Table 7).

*2. Stress and Relative Displacement Constraints, Two Load Condition*

The stress limits and the loading conditions are the same as those for the example problem C1. In addition to the stress limits, a limit of $\pm 1.0$ in. is put on the relative displacement along the $x$-direction between nodes 1 and 2 (Fig. 4). The present design ($W=6{,}646$ lb) is 8.5% heavier than the established optimum design[6] of 6,119 lb and has 16 active behavioral constraints (15 stress and the one relative displacement constraint) and two active minimum area constraints. This design was obtained in 20 iterations with $\beta=0.12$, $|\Delta|=0.08$. The results of the present method are not converged. The final design of Ref. 6 has 24 active behavioral constraints (23 stress and one relative displacement constraint) and nine active minimum area constraints.

Neither stress-limited or displacement-limited default design parameters were used because the problem is neither strictly stress limited nor predominantly displacement limited. Even though the relative displacement constraint is much like

Table 7 Iteration history for 63 bar wing carry through box

| Analysis no. | No. of active constraints | Weight (lb) |
|---|---|---|
| 1 | 2 | 30214.2 |
| 2 | 6 | 6387.0 |
| 3 | 9 | 5795.0 |
| 4 | 18 | 5428.3 |
| 5 | 11 | 5501.3 |
| 6 | 2 | 5667.0 |
| 7 | 33 | 5272.8 |
| 8 | 27 | 5286.9 |
| 9 | 6 | 5462.2 |
| 10 | 1 | 5747.3 |
| 11 | 44 | 5026.4 |

a stress constraint (both being a measure of the relative displacement of two nodal points) the fully stressing option to quickly locate minimum area bars cannot be used. The emphasis for this problem is not to duplicate previous results but to demonstrate that relative displacement constraints as well as stress and displacement constraints can be included in the present method.

## IV. Conclusions

An iterative optimality criteria based method has been presented for determining the minimum weight design of linear elastic redundant structures with constraints on member stresses, nodal displacements, and design variable sizes. The iterative resizing rule drives the design to a point in the design space satisfying the Kuhn-Tucker necessary conditions for a local optimum design. In the process, the subset of active constraints governing the final design is automatically identified by generating and updating a list of near-critical constraints at the beginning of each design cycle. The design factors of Eq. (13) are calculated and used to: 1) estimate the degree of convergence to the optimum design; and 2) modify the design variables if convergence is not satisfied. The results for several truss-type structures show that the present method is effective for generating near-optimum designs.

The present optimality criteria method is generally more complicated than previous physical optimality criteria methods, since it is necessary to identify the subset of active constraints, calculate the derivatives of the active constraints with respect to the design variables, and solve for the set of corresponding Lagrange multipliers. However, the present method converges to designs satisfying the Kuhn-Tucker necessary conditions for a local optimum design. If the iteration is terminated before the procedure converges the design factors can be used to estimate the degree of convergence.

The present method is generally more efficient than mathematical programing methods with respect to execution time and computer storage requirements. Large gains in efficiency are expected when the number of active constraints is smaller than the number of design variables. However, the results for the 63 bar wing carry-through box truss show the present method to be more efficient than the latest generation mathematical programing methods, even for problems with a large number of design variables and a large number of final active constraints.

Each redesign cycle of the present method requires a complete structural analysis (that is, decomposition of the stiffness matrix and reduction and back substitution of the load vectors) and iteration on the Lagrange multipliers within a redesign cycle requires a complete solution of the set of linear algebraic equations. Further and significant gains in computational efficiency can be obtained by incorporating approximation concepts[6,5] to reduce the number of complete structural analyses and by using iterative solution methods to reduce the number of complete solutions for the Lagrange multipliers.

It is hoped that the present approach, which is on the interface between conventional optimality criteria methods and mathematical programing methods, will prove useful not only in determining near-optimum preliminary designs, but will stimulate future efforts to develop efficient recursive design procedures based on the concepts of mathematical optimization.

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